Information Theory and Optimization of Analog Feedback Communication Systems

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***Abstract*—The paper is devoted to the analysis of connections between the limit accuracy of transmission and information characteristics for digital communication systems (DCS) and analog feedback CS (AFCS) applied to transmission of signals from analog sources. It is shown that the mean square error (MSE) of transmission determines information characteristics of AFCS. Differences between the capacities of AFCS considered as generalized communication channel and their forward channel are discussed. The new effects appearing in optimal AFCS and in DCS operating near Shannon's boundary are considered as well. The paper develops the results of the previous author's research and research in AFCS theory carried out in the years 1950-1970s.**

***Keywords*—analog signal transmission, adaptive modulation, information limits, feedback, power-bandwidth efficiency, capacity, accuracy of transmission, optimization, Bayesian estimation**

I. Introduction

In Shannon's first works [1],[2], digital and analog communication systems (CS) were considered without any preferences, and no proof that the application of analog CS is always less efficient than digital communication systems (DCS) had been given until now. Moreover, from 1960 to the middle of 1980, analog transmission was the subject of intensive wide-scale research ([3]–[12] and other works). The obtained results unambiguously proved the capability of analog feedback CS (AFCS) to transmit the signals without coding with a bit rate equal to the capacity of the forward channel. These results also determined the approach to designing corresponding "ideal" systems. The great interest in AFCS disappeared in the middle of 70th. Analysis of literature showed [13] that the reason was the lack of practical results after almost 20 years of work. This determined the digital future of communications and substantially hampered the development of AFCS theory. Nowadays, analog transmission is considered the past of communications.

Our research ([14],[15] and other works) in the optimization of adaptive feedback *estimation* systems (AFES) showed that information characteristics of optimal AFES attain the limit values, and solution of the optimization task enables the system design. Application of the approach developed in [14] ,[15] to AFCS [13],[16]–[18] allowed us to remove difficulties

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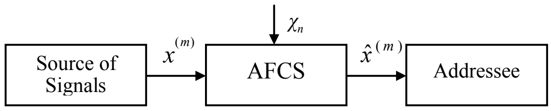

Fig. 1. AFCS as a generalized communication channel

that blocked implementation of the results of earlier research. The derived groups of relationships (optimal transmission–reception algorithm) enable a design of optimal AFCS transmitting the signals with minimal mean square error (MSE). It was established [15],[16] that optimal AFCS transmit the signals over a forward channel with the bit rate R^{Ch1} [bit/s] equal to its capacity C^{Ch1}, and power (E_n^{bit} / N_ξ [J/bit]) and bandwidth (R^{Ch1} / F_0 [bps/Hz]) efficiencies of transmission attain the theoretical limits. Moreover, the obtained relationships connect these characteristics with the parameters of the systems and scenario of application. The results of complex investigation of new effects and factors influencing the quality, bit rate, and efficiency of transmission show that AFCS theory is a practically untouched field for further theoretical and applied research.

The goal of this paper is to analyze connections between MSE of transmission and information characteristics of AFCS and DCS used for analog signals transmission. The differences between the capacities of AFCS and their forward channels are considered. It is shown that the rate distortion function is directly connected with the capacity of AFCS. The analysis is preceded by a discussion of the reasons of inapplicability of previous research in AFCS and ways to remove them. The presented results can be used to the wireless sensors design.

II. Completed Approach To AFCS Optimization

From a general point of view, every DCS and AFCS used for transmission of continuous signals is a data delivering channel of remote measurement systems or, more generally, of estimation systems (see Fig. 1). An analog source (sensor, microphone, etc.) generates signal x_t whose samples $x^{(m)}$, $m = 1, 2, ...$ are delivered by AFCS or DCS to the addressee in the form of estimates $\hat{x}^{(m)}$ of their values. These estimates recover the origin signal with errors caused by noises acting in the transmission channels. In metrology, estimation, and control theory, the basic measure of the accuracy of estimates is MSE or standard deviation of estimates. This is a clear and measurable criterion of the system's performance formulated based on mathematical models of the system's main components. It is worth noting that nowadays, information characteristics of estimation, control, or measurement systems cause no special interest. In turn, performance of DCS is evaluated by information characteristics but not by MSE independently from the continuous or digital signals transmitted. As in [5]–[12], MSE of the transmission is the basic criterion of AFCS performance.

A. Mathematical model of AFCS functioning

A general block diagram of AFCS is presented in Fig. 2. The input analog signal $x_t = x(t)$ is assumed to be a Gaussian stationary process with the mean value x_0, variance σ_0^2, and the baseband F_0 / F. The end node (EN) transmitter includes the sample-and-hold unit (S&H), subtractor (Σ), adaptive analog (AM) modulator ($\Sigma + \text{M1}$), and feedback receiver (R2). Each sample $x^{(m)} = x(m/2F)$ of input signal ($m = 1.2,...$) is held at the first input of the S&H unit during time $T = 1/2F$, permitting the system to transmit it in $n = T/\Delta t_0 = F_0 / F$ cycles. Value $\Delta t_0 = 1/2F_0$ is the duration of a single cycle of transmission, $2F_0$ is the bandwidth of forward channel Ch1, and bandwidth of the feedback channel Ch2 is not narrower than $2F_0$. It is assumed that each sample is transmitted in the same way and independently from previous samples. In this case, analysis of AFCS can be reduced to consideration of a single sample transmission, and index "*m*" in estimates $x^{(m)}$ can be omitted. We also assume forward and feedback channels are stationary and memoryless, and channel noises ξ_t, η_t are additive white Gaussian noises (AWGN).

At the first cycle of transmission, signal at the second input of subtractor Σ is set to the value x_0. In each of the next cycles, $k = 2,...,n$ signals at the second input of the subtractor are consequently replaced by estimates $\hat{x}_k^* = \hat{x}_{k-1} + \text{v}_k$ of the sample computed in the digital signal processing unit (DSP) of base station (BS) and delivered to the EN transmitter over feedback channel T2-Ch2-R2 (T2 denotes transmitter of BS). This channel can be digital or analog, and the only requirement is it should have sufficiently good characteristics for the variance of feedback transmission errors v_k caused by AWGN η_k satisfied the inequality: $\sigma_\text{v}^2 << \sigma_0^2$. In formulated conditions, independently from the method of feedback transmission, signal at the input of modulator M1 can be written as

$$e_k = x - \hat{x}_k^* = x - \hat{x}_{k\text{-}1} + \text{v}_k \,, \quad (\hat{x}_0 = x_0) \;. \tag{1}$$

As in [5]–[12], we assume modulator M1 is linear, and its modulation depth is set, for each k, to independently computed value M_k stored in the EN transmitter's memory unit. Then, the emitted signal s_k can be written as

$$s_{k,t} = A_0 M_k e_k \cos(2\pi f_0 t + \varphi_k) \tag{2}$$

where A_0 is the amplitude of the signal emitted to the channel under $M_k \,|\, e_k \,| = 1$, and f_0 and φ_k are its frequency and initial phase, respectively. Denoting the path-loss as γ and distance between the EN transmitter and BS as r, one can write the signal received by BS (in open space) as

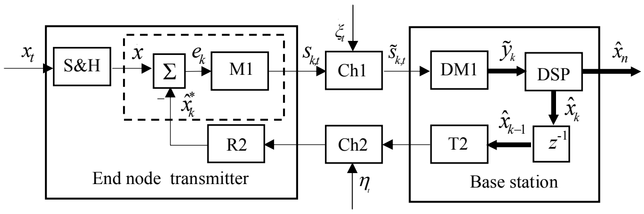

Fig. 2. Block diagram of AFCS

$$\tilde{s}_{k,t} = \frac{\gamma}{r} s_{k,t} + \xi_t \;. \tag{3}$$

Signal $\tilde{y}_k$ ("observation") formed by demodulator DM1

$$\tilde{y}_k = A M_k e_k + \xi_k \quad \text{where} \quad A = A_0 \frac{\gamma}{r} \tag{4}$$

is routed to the input of processing unit DSP. Variance σ_ξ^2 of noise ξ_k at the demodulator output is equal to the power of channel noise in the band $[f_0 - F_0, f_0 + F_0]$: $\sigma_\xi^2 = N_\xi F_0$ where $N_\xi / 2$ is the double-side spectral power density of AWGN ξ_t. Unit DSP computes estimates $\hat{x}_k = x_k(\tilde{y}_1^k)$ of the sample according to the Kalman-type equation:

$$\hat{x}_k = \hat{x}_{k-1} + L_k[\tilde{y}_k - E(\tilde{y}_k \,|\, \tilde{y}_1^{k-1})]; \;\; (\hat{x}_0 = x_0) \tag{5}$$

where $\tilde{y}_1^{k-1} \triangleq (\tilde{y}_1, ..., \tilde{y}_{k-1})$ denotes the sequence of observations received in previous cycles, and $E(\tilde{y}_k \,|\, \tilde{y}_1^{k-1})$ is the one-cycle prediction of the value of signal $\tilde{y}_k$ at the demodulator DM1 output. The gains L_k are computed additionally and stored in the memory of DSP unit. For each k, new estimate $\hat{x}_k$ replaces the previous one stored in the memory of DSP. Simultaneously, DSP routes it to the BS transmitter T2 and switches the gain L_k to value L_{k+1}. Signal $\hat{x}_{k-1} + v_k$ received by the EN transmitter is routed to the second input of subtractor Σ, and the gain M_k is set to the value M_{k+1}. This finalizes the cycle, and AFCS begins the next cycle of transmission. After n cycles, final estimate $\hat{x}_n$ is routed to addressee, and AFCS begins transmission of the next sample.

One can easily check that for each $k = 1,...,n$, mean values and correlations of the signals e_k, $\tilde{y}_k$ satisfy the relationships:

$$E(e_k \,|\, \tilde{y}_1^{k-1}) = 0\,; \quad E(e_i\, e_k \,|\, \tilde{y}_1^{k-1}) = (\sigma_\text{v}^2 + P_{k\text{-}1})\, \delta_{ik}\,; \tag{6}$$

$$E(\tilde{y}_k \,|\, \tilde{y}_1^{k-1}) = 0; \qquad E(\tilde{y}_i\, \tilde{y}_k \,|\, \tilde{y}_1^{k-1}) = [A^2 M_k^2(\sigma_\text{v}^2 + P_{k\text{-}1}) + \sigma_\xi^2]\, \delta_{ik} \tag{7}$$

(δ_{ik} is the Konecker symbol), and equation (5) takes the form:

$$\hat{x}_k = \hat{x}_{k-1} + L_k \tilde{y}_k\,; \;\; (\hat{x}_0 = x_0) \;. \tag{8}$$

Values P_k in (6),(7) describe the MSE of estimates formed by AFCS in sequential cycles: $P_k = E[(x - \hat{x}_k)^2]$, and $\hat{x}_k = E(x \,|\, \tilde{y}_1^k)$. One should note that according to known results of estimation theory, estimates $\hat{x}_k = E(x \,|\, \tilde{y}_1^k)$ minimize corresponding MSE.

B. Statistical fitting condition

Models (1) - (8) allow, for each k, direct computation of the MSE of transmission $P_k = E[(x - \hat{x}_k)^2]$, which becomes a function of free parameters $M_1,...,M_n$, $L_1,...,L_n$, and the values $M_1^{opt},...,M_m^{opt}$ and $L_1^{opt},...,L_n^{opt}$ minimizing MSE can be found. Similar optimization tasks were considered in [5]–[12] and other works but the obtained absolutely correct solutions were never used in practice. Analysis of the problem showed that the reason was a commonly used linear model of the EN transmitter. These models do not describe always possible saturation of the real transmitters appearing if the signals formed by modulator exceed the level A_0 determining output range of the emitter. Assuming that A_0 in (2) represents this range, adequate model of EN transmitter is to have the form:

$$s_{t,k} = A_0 \begin{cases} M_k e_k & \text{if } M_k |e_k| \le 1 \\ \text{sgn}(e_k) & \text{if } M_k |e_k| > 1 \end{cases} \cos(2\pi f_0 t + \varphi_k) \quad . \qquad (9)$$

Violation of inequality $M_k |e_k| \le 1$ in any cycle causes saturation of the transmitter, destruction of corresponding estimate $\hat{x}_n$ and loss of information about the sample. Even if saturations appear rarely, with a small probability $\mu << 1$ per cycle, they cause appearance, in average, of μ percent of distorted samples and bits in binary sequences at the output of AFCS. This means a probability of saturation μ can be considered as a direct analog of the bit error rate (BER) widely used in digital communications. Just omission of possible saturations made inapplicable the results of previous research in AFCS theory. Conventional methods of statistical linearization or direct constraints on the peak values or mean power of the emitted signals also do not solve the problem.

Nonlinear model (9) makes optimization of AFCS a non-Gaussian task, which can be solved only approximately. Moreover, MSE $P = E[(x-\hat{x})^2]$ is not sensitive to rare saturations of transmitters, so their influence on system performance can be adequately considered only as the additional condition in formulation of the optimization task. This condition was introduced in [14] and allowed us to obtain constructive results for different classes of systems with saturation [15]. The idea consists in reduction of the set of possible values of the gains M_k in (9) to the set Ω_k of "permissible" values "fitted" to the statistics of input signals in the way excluding, for each k, appearance of saturation with a probability not greater than given small μ, ($\mu \le 10^{-5}$). Setting the gains M_k to these values practically excludes appearance of saturation during the sample transmission. This permits us to replace nonlinear model (9) with linear model (2), which recovers a possibility to optimize AFCS using the methods of Bayesian optimal estimation theory. MSE P_k of estimates formed by nonlinear and linear AFCS may differ by the values of $O(\mu)$ order, which determine the accuracy of the obtained solutions.

Analytically, the discussed constraint on the set of M_k values ("statistical fitting condition" [14],[15]) can be formulated as follows: for each $k = 1,...,n$, permissible values M_k should satisfy the inequality

$$\Pr_k^{sat} = \Pr(M_k \, |e_k| > 1 \big| \, \tilde{y}_1^{k-1}) = 1 - \int_{-1/M_k}^{1/M_k} p(e_k \mid \tilde{y}_1^{k-1}) \, dx < \mu \qquad (10)$$

where $p(e_k \mid \tilde{y}_1^{k-1})$ is the posterior distribution of the signals $e_k = x - \hat{x}_{k-1} + v_k$. The explicit form for this distribution can be easily found using formulas (6):

$$p(e_k \mid \tilde{y}_1^{k-1}) = \frac{1}{\sqrt{2\pi(\sigma_v^2 + P_{k-1})}} \exp\left(-\frac{e_k^2}{2(\sigma_v^2 + P_{k-1})} \right). \qquad (11)$$

Substitution of (11) into (10) gives the relationships (e.g [13]):

$$\Phi(\alpha) = \frac{1}{\sqrt{2\pi}} \int_0^{\alpha} e^{-\frac{x^2}{2}} dx \ \ge \ \frac{1}{2}(1-\mu) \ ; \ \alpha = \frac{1}{M_k \sqrt{\sigma_v^2 + P_{k-1}}} \qquad (12)$$

where $\Phi(\alpha)$ is tabularized Gaussian integral. Formula (12) determines the set Ω_k of permissible, for each k, values:

$$\Omega_k = \left\{ M_k : \ 0 < M_k \le M_k^{\max} = \frac{1}{\alpha \sqrt{\sigma_v^2 + P_{k-1}}} \right\}, \qquad (13)$$

where α is determined by the equation $\Phi(\alpha) = (1-\mu)/2$.

Corollary 1. Adequate optimization of AFCS should be carried out under additional ("fitting") condition (10), and performance of AFCS is to be assessed by two criteria: a) MSE of transmission and b) probability of saturation (BER) μ.

C. Optimal transmission-reception algorithm

The optimization task consists in definition of the values $M_1^{opt},...,M_m^{opt}$ and $L_1^{opt},...,L_n^{opt}$ minimizing MSE of estimates at the AFCS output under additional constraint (13). This task can be solved in different ways (e.g. [13],[16],[18]), and the derived optimal transmission-reception algorithm is as follows. The EN transmitter operates according to relationships (1),(2) where gains M_k are set to the values

$$M_1^{opt} = \frac{1}{\alpha\sigma_0}; \ M_k^{opt} = M_k^{\max} = \frac{1}{\alpha\sqrt{\sigma_v^2 + P_{k-1}^{min}}}, \ k = 2,...,n. \ (14)$$

Optimal estimates $\hat{x}_k^{opt}$ are computed in BS according to (8) under gains L_k set to the values

$$L^{opt} = \frac{A M_k P_{k-1}^{\min}}{\sigma_\xi^2 + A^2 M_k^2 (\sigma_v^2 + P_{k-1}^{\min})} = \frac{1}{A M_k^{opt}} \left(1 - \frac{P_{k-1}^{\min}}{P_k^{\min}} \right). \qquad (15)$$

Parameters $P_k^{\min}$ in (14), (15) are values of minimal MSE (MMSE) of estimates achieved after k cycles of transmission and satisfying the recursive equation of Riccati type:

$$P_k^{\min} = \frac{(1+Q^2)\sigma_v^2 + P_{k-1}^{\min}}{(1+Q^2)(\sigma_v^2 + P^{\min})} P_k^{\min}, \ (P_0^{\min} = \sigma_0^2) \qquad (16)$$

where

$$Q^2 = \frac{W_{\max}^{Ch1}}{\sigma_\xi^2} = \left(\frac{A}{\alpha\sigma_\xi^2} \right) = \left(\frac{A_0 \gamma}{\alpha r} \right) \frac{1}{N_\xi F_0} = SNR_{out}^{Ch1} \qquad (17)$$

is maximal signal-to-noise ratio (SNR) at the channel Ch1 (demodulator DM1) output, and $W_{\max}^{Ch1} = A^2 (M_k^{opt})^2 E(e_k^2) = (A/\alpha)^2$ is maximal, under given μ, power of information component in the received signal at the DM1 output. The presented relationships contain all necessary information permitting to design optimal AFCS [19].

III. Information Characteristics For Optimal AFCS

To make the relationships more readable, except from necessary cases, indices "opt", "min", "max" are omitted.

A. Information characteristics for the forward channel

In the considered communication scheme (Fig. 2), the forward cannel includes modulator M1, channel Ch1, and demodulator DM1, and saturation unit Σ is the source of signals.

Using formulas (4),(6),(7), one can easily find conditional entropies of Gaussian signals $\hat{y}_k$ and e_k:

$$H(\tilde{Y}_k \mid E_k, \tilde{Y}_1^{k-1}) = \frac{1}{2}\log_2(2\pi e\,\sigma_\xi^2), \tag{18}$$

$$H(\tilde{Y}_k \mid \tilde{Y}_1^{k-1}) = \frac{1}{2}\log_2 2\pi e\,[\sigma_\xi^2 + A^2 M_k^2(\sigma_x + P_{k-1})] = = \frac{1}{2}\log_2 2\pi e\left(\sigma_\xi^2 + \frac{A^2}{\alpha^2}\right) \tag{19}$$

where variables E_k, $\hat{Y}_k$ denote the sets of estimates of the input e_k and received $\tilde{y}_k$ signals, respectively. Then, for each k, and gains M_k set to the values (14), the amount of information in received signals about the input signals is determined by the formula:

$$I(\tilde{Y}_k, E_k \mid \tilde{Y}_1^{k-1}) = \frac{1}{2}\log_2\left(1+\frac{A^2}{\alpha^2\sigma_\xi^2}\right) = \frac{1}{2}\log_2\left(1+\frac{W_{\max}^{Ch1}}{N_\xi F_0}\right). \tag{20}$$

Considering duration of a single cycle is $\Delta t_0 = 1/2F_0$, formula (20) permits us to find the instant bit rate of transmission over the channel M1-Ch1-DM1:

$$R_k^{Ch1} = \frac{I(\tilde{Y}_k, E_k \mid \tilde{Y}_1^{k-1})}{\Delta t_0} = F_0 \log_2\left(1+\frac{W_{\max}^{Ch1}}{N_\xi F_0}\right) = F_0 \log_2(1+Q^2). \tag{21}$$

In optimal AFCS, power $W_{\max}^{Ch1}$ of useful signal at the DM1 output takes, for every n and $1 \le k \le n$, constant and maximal permissible value. For every non-optimal AFCS, bit rate will be lower than bit rate (21). This permits us to conclude:

Corollary 2. Independently from the number of cycles, optimal statistically fitted AFCS designed on the basis of formulas (1),(2),(4),(8),(14)–(17) transmit the signals over the forward channel with maximal available, under given μ, bit rate (21) ,which defines the capacity of the channel:

$$C^{Ch1} = F_0 \log_2\left(1+\frac{W_{\max}^{Ch1}}{N_\xi F_0}\right) = F_0 \log_2(1+Q^2) \text{ [bit/s]}. \tag{22}$$

Formulas (21),(22) have the form identical to the classic formula for the capacity of the channels with AWGN. However, these formulas are obtained as result of minimization of the MSE of transmission, but not as the extreme of the amount of mutual information on the set of prior distributions [1],[2]. It is important to stress that, unlike the classic result, power of the received signals $W_{\max}^{Ch1}$, SNR Q^2 and capacity (22) directly depend on the saturation factor $\alpha = \alpha(\mu)$ which, in turn, depends on the given value μ (BER). The smaller μ, the greater α and lesser capacity of the channel and vice versa. Dependence of SNR Q^2 on γ, r, and N_ξ makes channel capacity dependent on the scenario of AFCS system application.

The limit power-bandwidth (P-B) efficiency of transmission over the forward channel satisfies the relationship directly following from (22) and coinciding with well-known result:

$$\frac{E^{bit}}{N_\xi} = \frac{F_0}{C^{Ch1}}\left(2^{\frac{C^{Ch1}}{F_0}} - 1\right) = \frac{Q^2}{\log_2(1+Q^2)}, \tag{23}$$

where C^{Ch1}/F_0 describes limit B-efficiency of the channel.

B. Information characteristics for AFCS as whole

For every statistically fitted Gaussian AFCS that forms estimates $\hat{x}_k$ with MSE P_k, ($P_n >> \sigma_v^2$, $k = 1,...,n$), mutual amount of information between the samples and their estimates is determined by the relationship

$$I(X, \hat{X}_k) = H(X) - H(X \mid \hat{X}_k) = \frac{1}{2}\log_2\frac{\sigma_0^2}{P_k}. \tag{24}$$

If the samples are transmitted in n cycles, and duration of the cycle is $\Delta t_0 = 1/2F_0$ than, according to (24), mean bit rate at the output of AFCS is equal to

$$\overline{R}_n^{AFCS} = \frac{I(X, \hat{X}_n)}{T_n} = F_n \log_2\frac{\sigma_0^2}{P_n} = \frac{F_0}{n}\log_2\frac{\sigma_0^2}{P_n}\ ;\ (T_n = n/2F_0). \tag{25}$$

Formula (24) has the form of Gaussian rate distortion function [7],[11],[12], and (25) determines the minimal bit rate permitting AFCS to reconstruct samples with a given accuracy P_k. As it was shown above, optimal AFCS designed according to (1),(2),(4),(8),(14)–(17) transmit the samples with MSE $P_k^{\min}$ minimal on the set of possible values of MSE provided by the set of AFCS with different permissible parameters $M_k \in \Omega_k$, (it is assumed that power of the transmitter, baseband of signals, bandwidth, and scenario of application are the same). Thus, bit rate (25) is the upper boundary of bit rates provided by different versions of AFCS designed and used in the same conditions. The latter allows us to consider this boundary as the capacity C_n^{AFCS} of AFCS considered as a generalized communication channel (Fig. 1):

$$C_n^{AFCS} = F_n \log_2\frac{\sigma_0^2}{P_n^{\min}} = \frac{F_0}{n}\log_2\frac{\sigma_0^2}{P_n^{\min}} \tag{26}$$

where values of MSE $P_k^{\min}$ are determined by formula (16).

Corollary 3. The bit rate $\overline{R}_n^{AFCS}$ of any statistically fitted AFCS with the same parameters A_0, F_0, $\alpha = \alpha(\mu)$ and used in the same scenario satisfies the inequality

$$\overline{R}_k^{AFCS} \le C_k^{AFCS}. \tag{27}$$

Transmission with a bit rate greater that C_n^{AFCS} violates the statistical fitting condition (10) that is increases a probability of AFCS saturation to unacceptably large values.

Corollary 4. To transmit signals with a bit rate equal to the capacity (22), AFCS should be designed in the way diminishing errors of the signals recovering until the limit values determined by formulas (16),(17). These values should be assessed prior to the application of AFCS in the given scenario.

C. Differences between characteristics of channel and AFCS

For small power EN transmitters and sufficiently powerful BS transmitters, the following inequality is valid:

$$SNR_{inp}^{M1} = \frac{\sigma_0^2}{\sigma_v^2} >> 1 + Q^2 = 1 + SNR_{out}^{Ch1} \tag{28}$$

where $SNR^{inp} = \sigma_0^2/\sigma_v^2$ is SNR at the modulator M1 input. In this case, for a sufficiently large *n*, interval $[1, n]$ decays into two sub-intervals. At the "pre-threshold" interval, $1 \le k \le n^*$, $(n^* < n)$ MMSE P_k satisfies the inequality $P_k >> \sigma_v^2(1+Q^2)$ and $P_k << \sigma_v^2(1+Q^2)$ at the "post-threshold" interval $n^* < k \le n$.

The threshold point $n*$ is the solution of the equation $P_n = \sigma_v^2$ and is determined by the relationship:

$$n* = \frac{1}{\log_2(1+Q^2)} \quad \log_2\left(\frac{\sigma_0^2}{\sigma_v^2}\right) = \frac{\log_2(SNR_{inp}^{Ch1})}{\log_2(1+SNR_{out}^{Ch1})} \qquad (29)$$

and (16) can be replaced by the approximate relationship:

$$P_k^{\min} = \begin{cases} \sigma_0^2(1+Q^2)^{-k} & \text{for } 1 \le k \le n* \\ \sigma_v^2(k-n*+1)^{-k} & \text{for } k > n* \end{cases} . \qquad (30)$$

For $k=n$, substitution of (30) into (27) gives the explicit expression for the AFCS capacity:

$$C_n^{AFCS} = \begin{cases} F_0 \log_2(1+Q^2) & \text{for } 1 \le n \le n* \\ \frac{F_0}{n} \log_2\left[\frac{\sigma_0^2}{\sigma_v^2}(n-n*+1)\right] & \text{for } n > n* \end{cases} . \qquad (31)$$

Corollary 5. According to (30), at the interval $1 \le n \le n*$, capacity of AFCS is equal to the capacity of the forward channel (22) independently from the characteristics of the feedback channel that agrees with known results of Shannon [4]. The same concerns the limit B-efficiency of C_n^{AFCS} / F_0 and P-efficiency of AFCS determined by the relationship

$$\frac{E_n^{bit\ AFCS}}{N_\xi} = \frac{Wn\Delta t_0}{N_\xi I(X,\hat{X}_n)} = \frac{nW}{2N_\xi F_0 I(X,\hat{X}_n)} = \frac{nQ^2}{\log_2 \frac{\sigma_0^2}{P_n^{\min}}} . \quad (32)$$

For $n > n*$, limit P-B efficiency of AFCS falls down (see Fig. 3), while MSE of transmission slowly tends to zero.

This effect can be explained as follows. According to (29), for the noiseless feedback channels ($\sigma_v^2 = 0$), $n^*=\infty$ and relationships (26),(32) coincide with (22), (23) for every $1 \le k \le n* < \infty$. For the noisy feedback and fulfilled (28), at the interval $1 \le k \le n*$, MSE satisfies the inequality $P_n >> \sigma_v^2$, and feedback errors practically do not influence transmission. In turn, for $k > n*$, $\sigma_v^2 >> P_k^{mln}$ and, according to (14), gains M_k attain the values of $1/\alpha\sigma_v$ order, which reduces the power of noise ξ_k in observations $\tilde{y}_k$ until the level of feedback noise v_k : $\tilde{y}_k = AM_k e_k + \xi_k = AM_k(x - \hat{x}_{k\text{-}1} + \text{v}_k + \alpha\sigma_v\xi_k)$. For this reason, at the interval $n* < k \le n$, the forward channel works as a *de facto* ideal noiseless channel, and DSP unit reconstructs estimates $\hat{x}_k$ of the sample, processing small useful signals $x - \hat{x}_{k\text{-}1}$ in the noise $\text{v}_k + \alpha\sigma_v\xi_k$. This slows down the rate of MSE diminution and decreases the capacity and P-B efficiency of AFCS. The bit rate and P-B efficiency of transmission over the forward channel are maintained at the level (22),(23) independently from the number of cycles.

IV. CONCLUSIONS

The presented results show that MSE is the basic performance criterion of AFCS performance, and optimal AFCS transmit signals with limit accuracy, bit rate, and P-B efficiencies. Feedback noise causes declination of AFCS information characteristics from boundaries (22),(23). This effect should also appear in DCS, whose performance approaches these boundaries. The presented transmission-reception algorithm enables designing P-B efficient channels for wireless sensors.

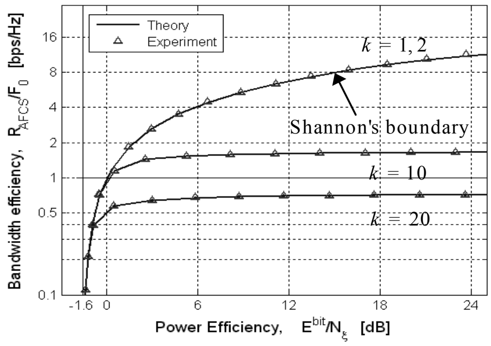

Fig. 3. Changes of power-bandwidth efficiency of optimal AFCS depending on the number of transmission cycles (results of computer experiment)

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